HIDDEN TOPOLOGICAL FEATURES OF PLANAR ISING NETWORK

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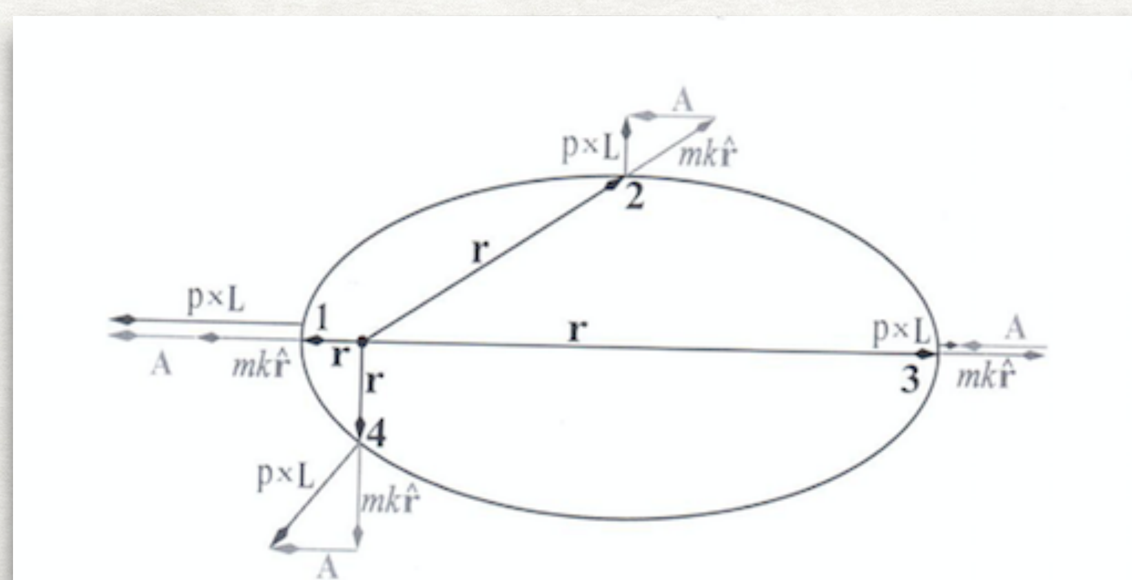
TO THE HIDDEN SIDE

In the past, we've seen that even when we have the correct description of a system, in that we can give precise predictions for the physical observables, we may still be entirely missing the fundamental nature of the system entirely!

Exp: The Kepler problem. Rotation invariance of the potential predicts three conserved quantities -> The normal direction of the rotation plane

In principle it can precess. But the fact that it doesn't, implies that there are new conservation quantities -> associated with the direction of the long axes



The Laplace-Range-Lenz vector

$$\vec{A} = \vec{P} \times \vec{L} - mk\hat{r}$$

There is a hidden SO(4)

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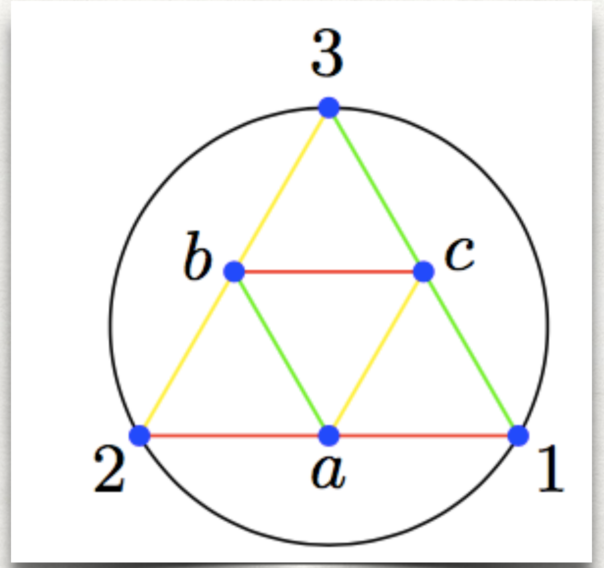
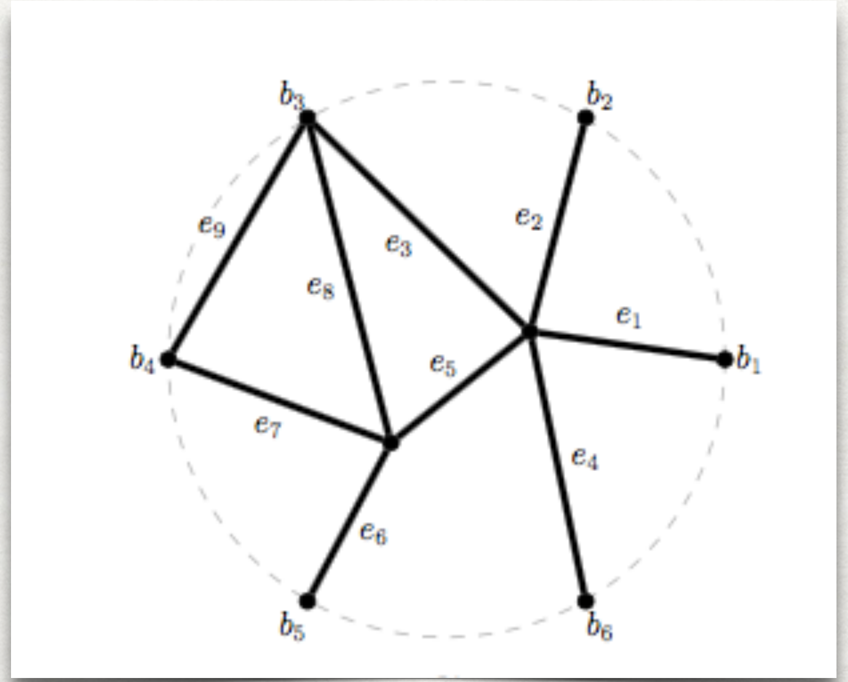
Lessons:

- Don't just shut up and calculate, observe the observable
- Look out for features that are not manifest in the underlying formulation: potential new understanding and computation power.

TOTAL POSITIVITY IN ISING

Let us consider planar Ising networks with n boundary sites and arbitrary coupling $J_{ab}>0$





Consider the two point function of planar Ising network

$$\langle \sigma_i \sigma_j \rangle = \frac{\sum_{\sigma_a \in \{\pm 1\}} \sigma_i \sigma_j P(J_{ab})}{\sum_{\sigma_a \in \{\pm 1\}} P(J_{ab})}, \quad P(J_{ab}) = \prod_{a,b \in \{E\}} e^{J_{ab}\sigma_a \sigma_b}$$

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Since $J_{ab}>0$ we know that $<\sigma_i \sigma_j> >0$

But in reality:

-0.00123+0.00257+0.01206-0.00786+……… >0

The positivity of the correlation is far from obvious

TOTAL POSITIVITY IN ISING

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Not only is $\langle \sigma_i \sigma_j \rangle > 0$

$$\begin{pmatrix} 1 & \langle \sigma_1 \sigma_2 \rangle & \langle \sigma_1 \sigma_3 \rangle & \cdots \\ \langle \sigma_2 \sigma_1 \rangle & 1 & \langle \sigma_2 \sigma_3 \rangle & \cdots \\ \langle \sigma_3 \sigma_1 \rangle & \langle \sigma_3 \sigma_2 \rangle & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

The n x n matrix of $\langle \sigma_i \sigma_j \rangle$ has all positive minors !

The matrix has total positivity

TOTAL POSITIVITY IN ISING

$$\langle \sigma_i \sigma_j \rangle = \frac{\sum_{\sigma_a \in \{\pm 1\}} \sigma_i \sigma_j P(J_{ab})}{\sum_{\sigma_a \in \{\pm 1\}} P(J_{ab})}, \; P(J_{ab}) = \prod_{a,b \in \{E\}} e^{J_{ab}\sigma_a \sigma_b}$$

P. Galashin and P. Pylyavskyy showed that when embedded in nx2n matrix

$$\begin{pmatrix} \overset{c_1}{1} & \overset{c_2}{1} & \overset{c_3}{\langle \sigma_1 \sigma_2 \rangle} & \overset{c_a}{-\langle \sigma_1 \sigma_2 \rangle} & \overset{c_b}{-\langle \sigma_1 \sigma_3 \rangle} & \overset{c_4}{\langle \sigma_1 \sigma_3 \rangle} \\ -\langle \sigma_1 \sigma_2 \rangle & \langle \sigma_1 \sigma_2 \rangle & 1 & 1 & \langle \sigma_2 \sigma_3 \rangle & -\langle \sigma_2 \sigma_3 \rangle \\ \langle \sigma_1 \sigma_3 \rangle & -\langle \sigma_1 \sigma_3 \rangle & -\langle \sigma_2 \sigma_3 \rangle & \langle \sigma_2 \sigma_3 \rangle & 1 & 1 \end{pmatrix}$$

Each row has the property that it is mutually null with respect to (+, -, +, -, …..) signature

$$\langle \sigma_i \sigma_j \rangle = \frac{\sum_{I \in \varepsilon(\{i,j\})} \Delta_I}{\sum_{I \in \varepsilon(\{\emptyset\})} \Delta_I}$$

We get back the correlation function through ratios of the minors. Each minor is positive

TOTAL POSITIVITY IN ISING

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In other words the two-point function of an Ising network has an image as an nx2n matrix modulo GL(n): the moduli space of n null plane in 2n dimensions, the Orthogonal Grassmannian

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & M_{26} \\ M_{31} & M_{32} & M_{33} & M_{34} & M_{35} & M_{36} \end{pmatrix} = \begin{pmatrix} \leftarrow \vec{n}_1 \rightarrow \\ \leftarrow \vec{n}_2 \rightarrow \\ \leftarrow \vec{n}_3 \rightarrow \end{pmatrix}$$

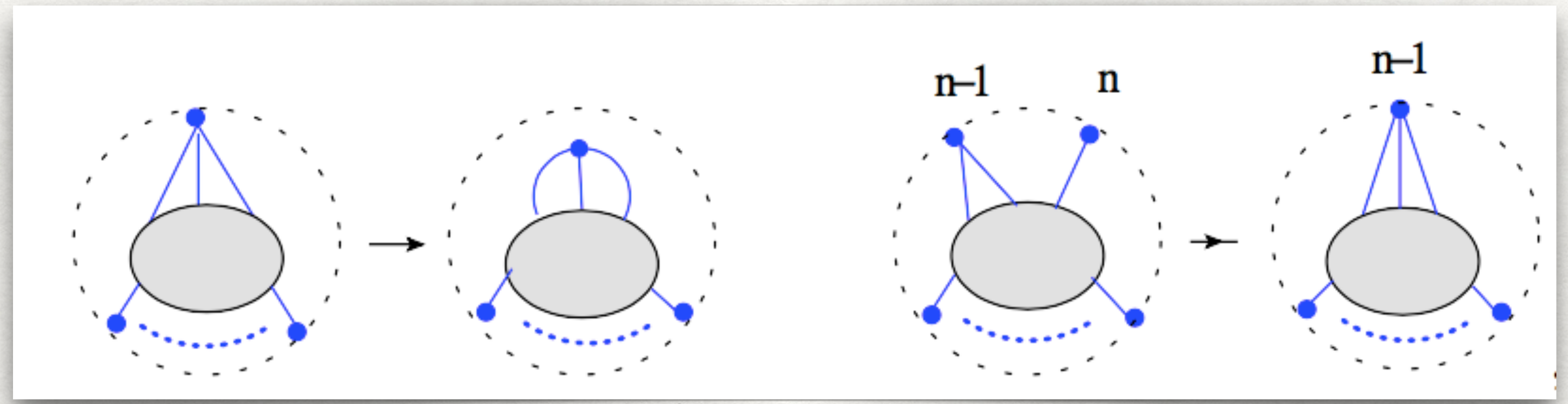
Galashin and P. Pylyavskyy tells us that it is positive Orthogonal Grassmannian !

TOTAL POSITIVITY IN ISING

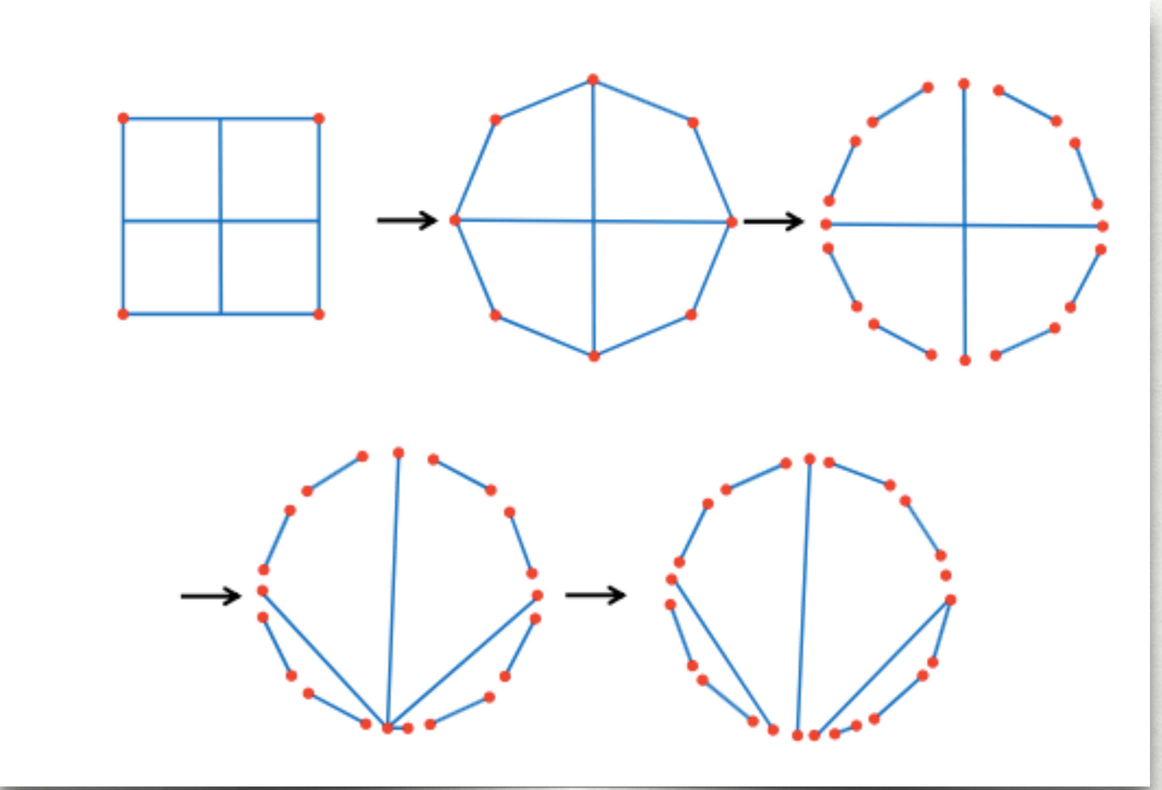
But why ?????

Note that all Ising networks can be constructed through amalgamations from free-edge networks

- Two fundamental steps for amalgamation



- Through inverse of these steps any network can be reduced to trivial free-edge network



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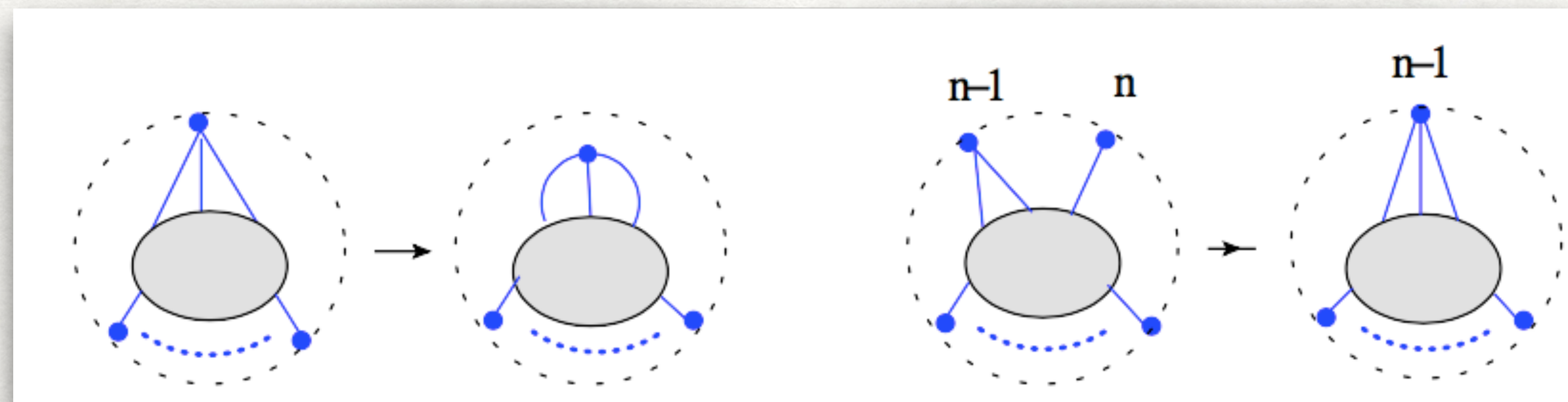
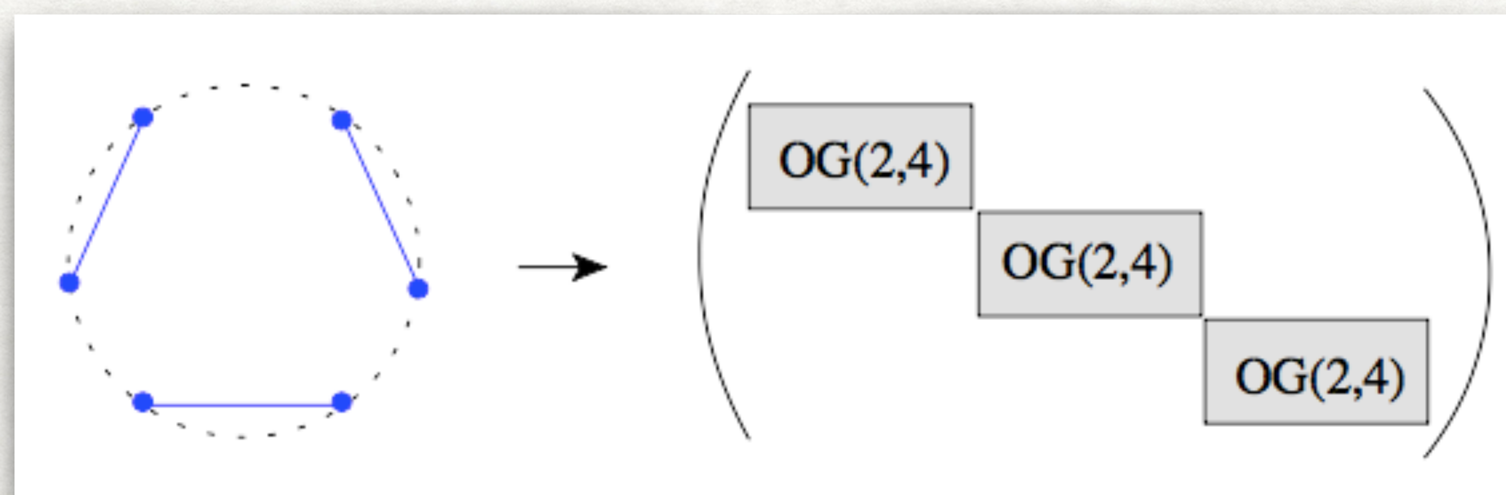


- For free edge networks the positivity is straight forward



If amalgamation preserves positivity, then the relationship is established!

TOTAL POSITIVITY IN ISING

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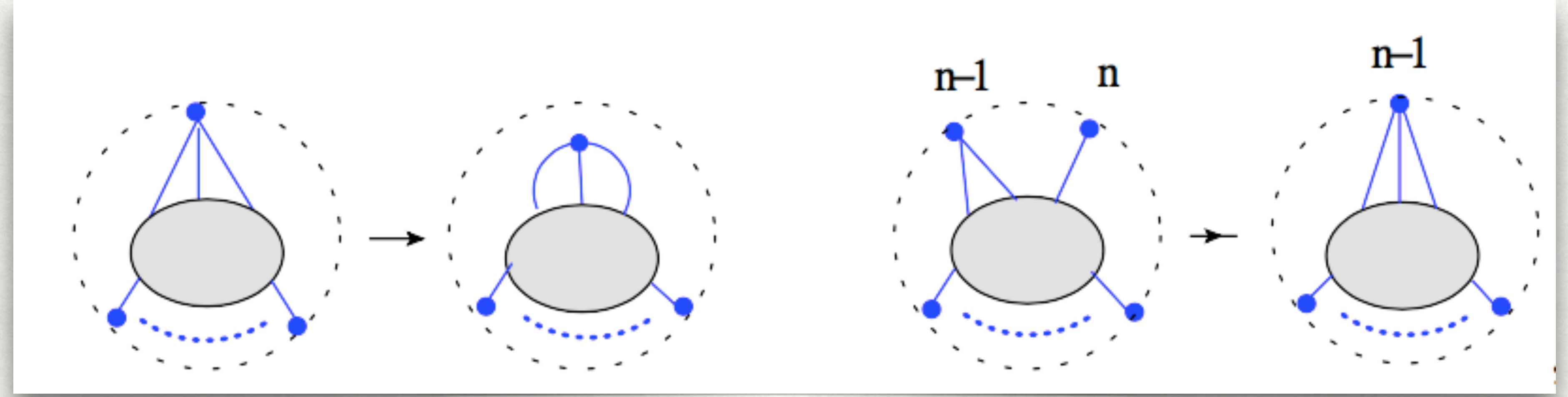
Note that all Ising networks can be constructed through the amalgamation of free edges



The operation of amalgamation translate to a non-linear identity for the two point function

$$\langle\sigma_i\sigma_j\rangle^{amal} = \frac{\langle\sigma_i\sigma_j\rangle + \langle\sigma_n\sigma_{n-1}\sigma_i\sigma_j\rangle}{1+\langle\sigma_n\sigma_{n-1}\rangle}$$

as we identify the two external spins, we are essentially subtracting $\sigma_n = -\sigma_{n-1}$ from Σ

$$\frac{\sum_{\sigma_a\in\{\pm1\}}\sigma_i\sigma_j P(J_{ab}) + \sum_{\sigma_a\in\{\pm1\}}\sigma_n\sigma_{n-1}\sigma_i\sigma_j P(J_{ab})}{\sum_{\sigma_a\in\{\pm1\}} P(J_{ab}) + \sum_{\sigma_a\in\{\pm1\}}\sigma_n\sigma_{n-1} P(J_{ab})}$$

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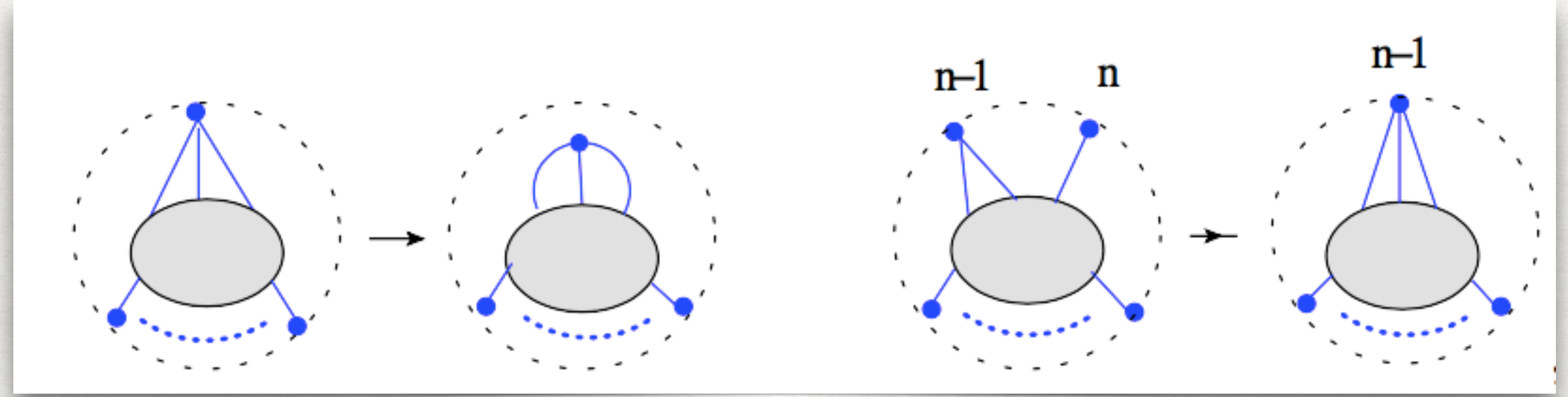

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But when translated in to minors of the nx2n matrix it is just a SUM!

$$\Delta^{OG(n-1,2n-2)}_{\{I\}} = \Delta^{OG(n,2n)}_{\{Ia\}} + \Delta^{OG(n,2n)}_{\{Ib\}}$$

If the trivial Ising network is positive, so is everyone!

TOTAL POSITIVITY IN ISING

But why ?????

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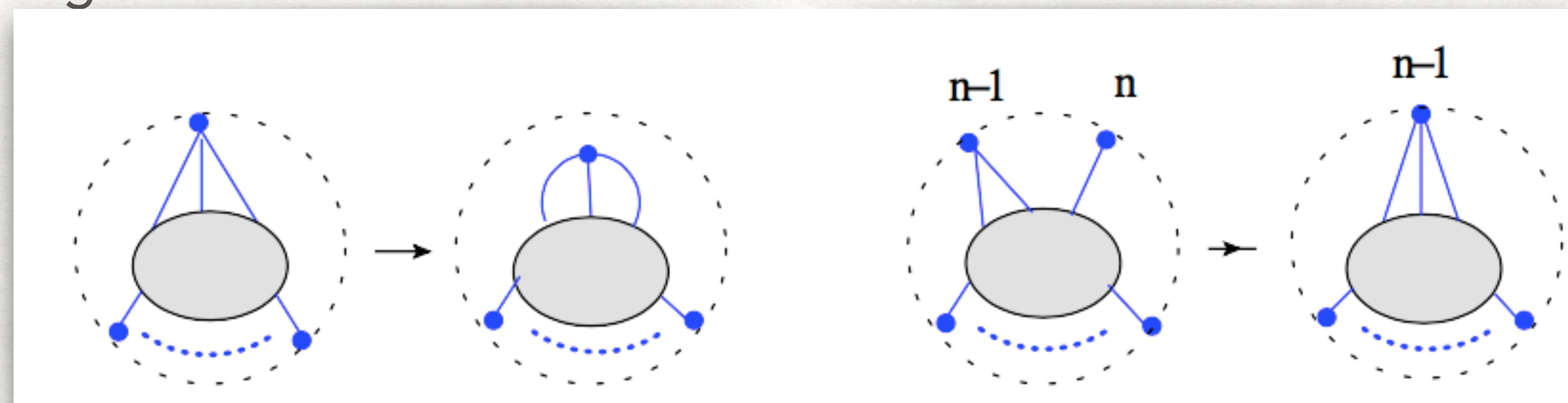

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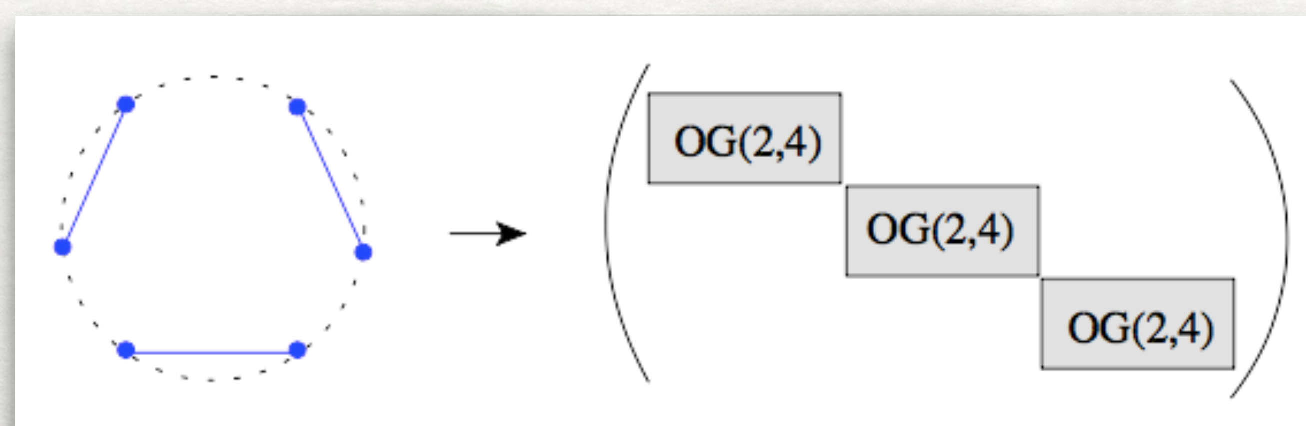


$$\begin{pmatrix} 1 & s(J) & 0 & -c(J) \\ 0 & c(J) & 1 & s(J) \end{pmatrix}$$

where

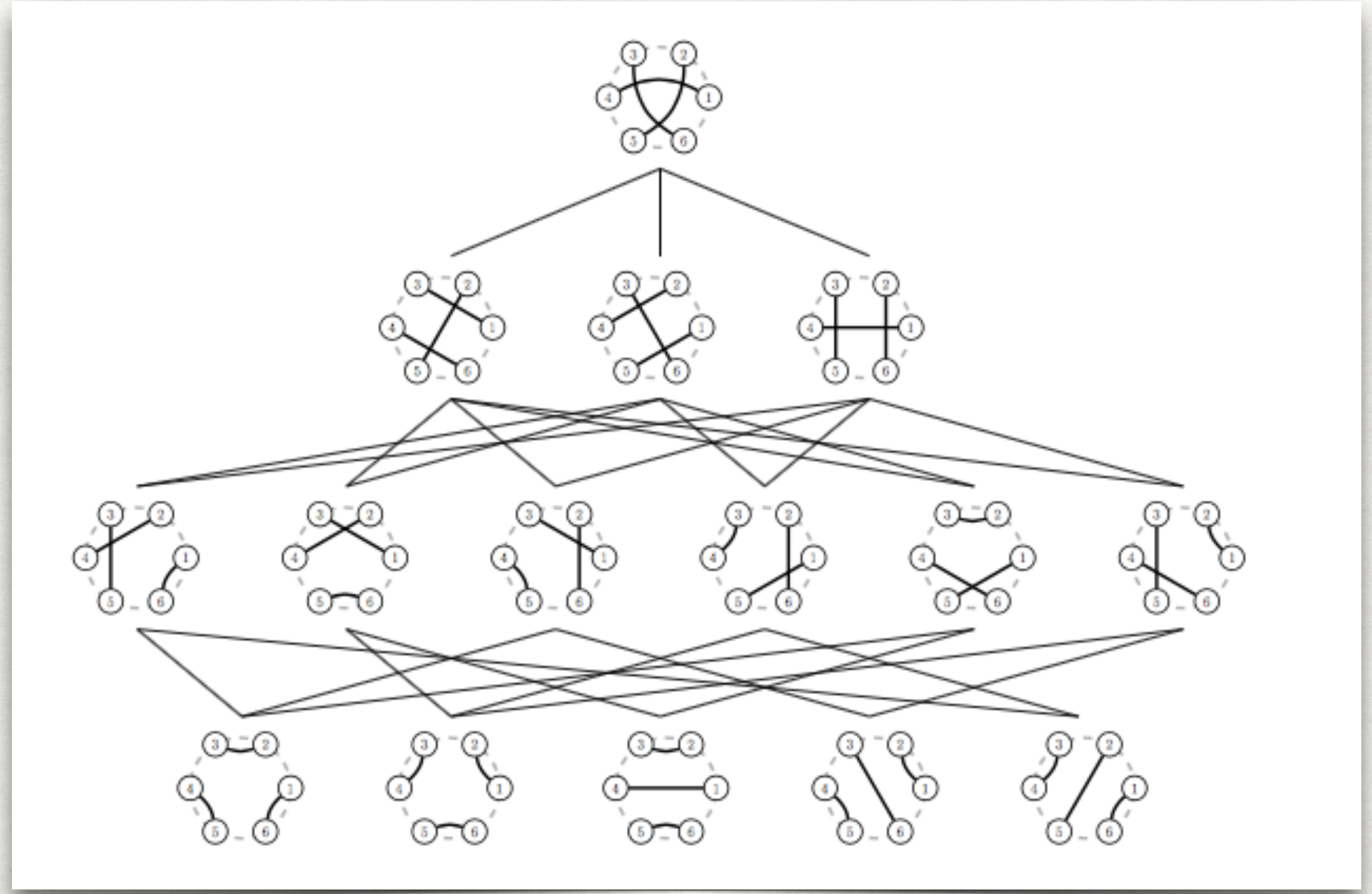
$$s(J) = \frac{2}{e^{2J}+e^{-2J}}, \qquad c(J) = \frac{e^{2J}-e^{-2J}}{e^{2J}+e^{-2J}}.$$

TOTAL POSITIVITY IN ISING

The space of total positive matrices mod GL(n) is finite. It is given by a stratification

The vanishing of the minors define a topologically distinct categorization

They form a topological ball



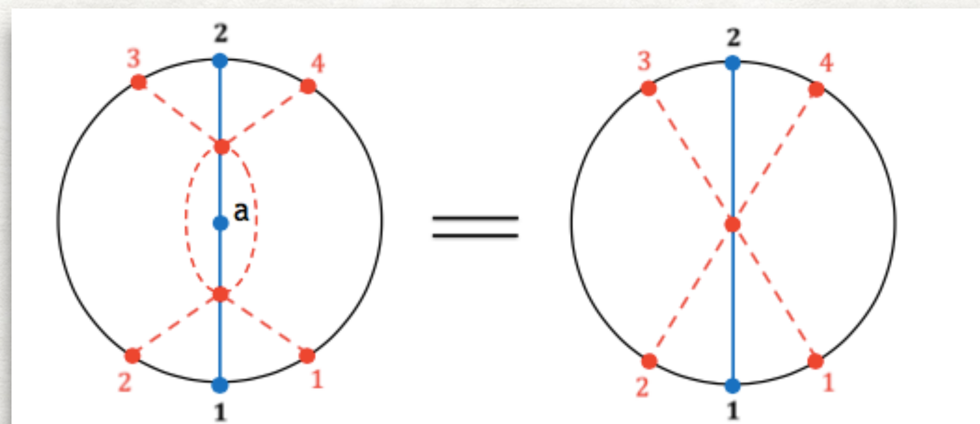
But the number of Ising networks is infinite!

TOTAL POSITIVITY IN ISING

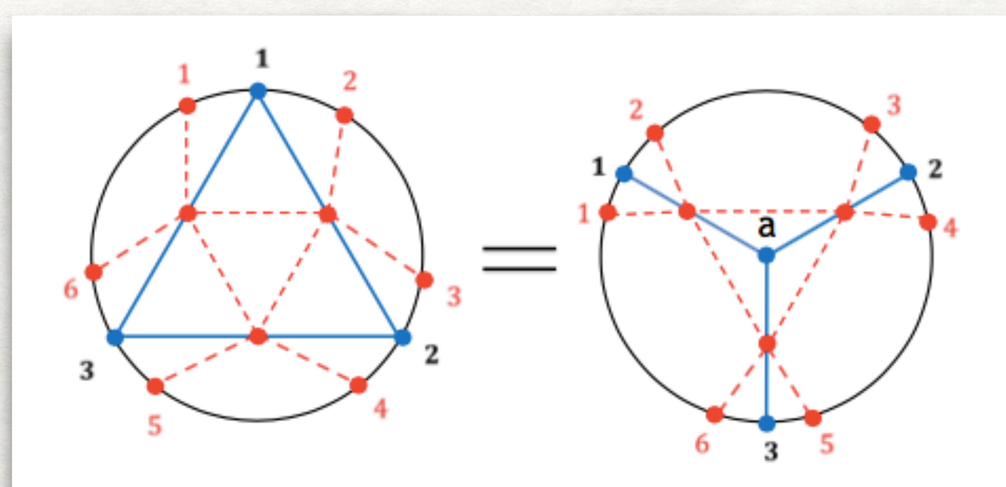
The space of total positive matrices mod GL(n) is finite

But the number of Ising networks is infinite!

Ising networks are secretly dual to each other though local duality moves !



$$c(J_{12}) = \frac{c(J_{1a})c(J_{2a})}{1+s(J_{1a})s(J_{2a})},$$
$$s(J_{12}) = \frac{s(J_{1a})+s(J_{2a})}{1++s(J_{1a})s(J_{2a})}$$

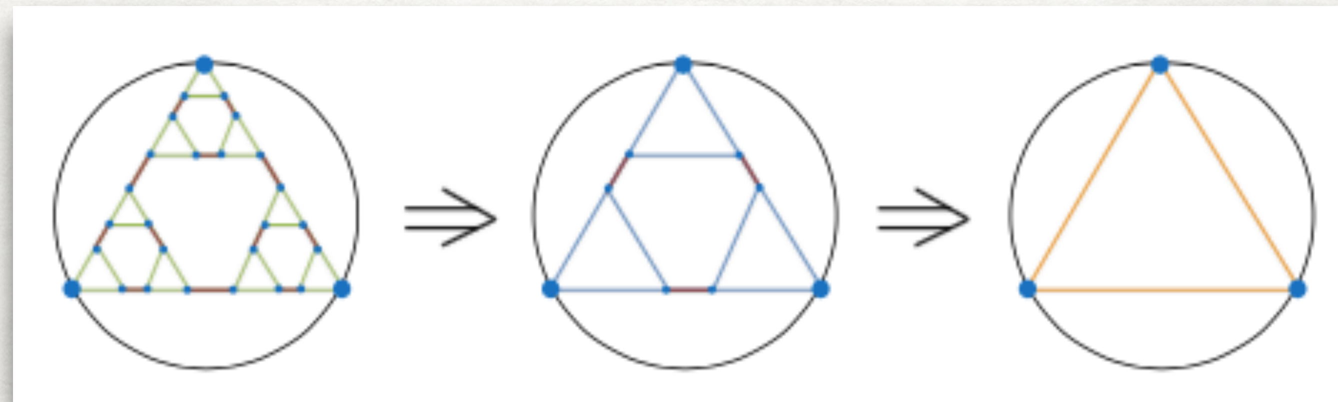


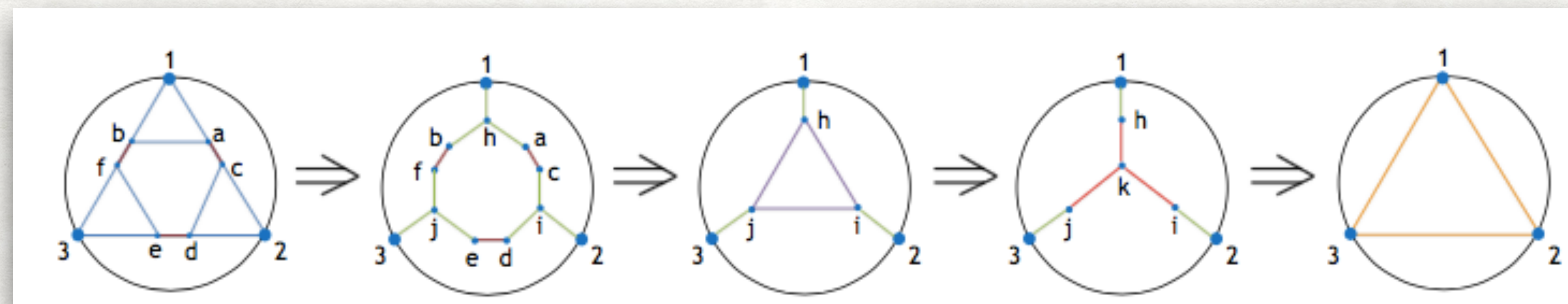
$$s(J_{ia}) = \frac{s(J_{ii+1})c(J_{i+1\,i+2})s(J_{i\,i+2})}{c(J_{i+1\,i+2})+c(J_{ii+1})c(J_{i\,i+2})},$$
$$c(J_{ii+1}) = \frac{c(J_{ia})c(J_{i+1\,a})s(J_{i+2\,a})}{s(J_{i+2\,a})+s(J_{ia})s(J_{i+1\,a})},$$

$$s(J) = \frac{2}{e^{2J}+e^{-2J}}, \qquad c(J) = \frac{e^{2J}-e^{-2J}}{e^{2J}+e^{-2J}}.$$

TOTAL POSITIVITY IN ISING

For self-repeating lattices (Fractals) this leads to recursion relation for the effective coupling





$$c(J_2') = \frac{c(J_1)\,c(J_2)^2}{c(J_2)^2 + 2(1 - c(J_2))(1 + s(J_1) - \frac{1}{2}c(J_1)\,c(J_2))}$$

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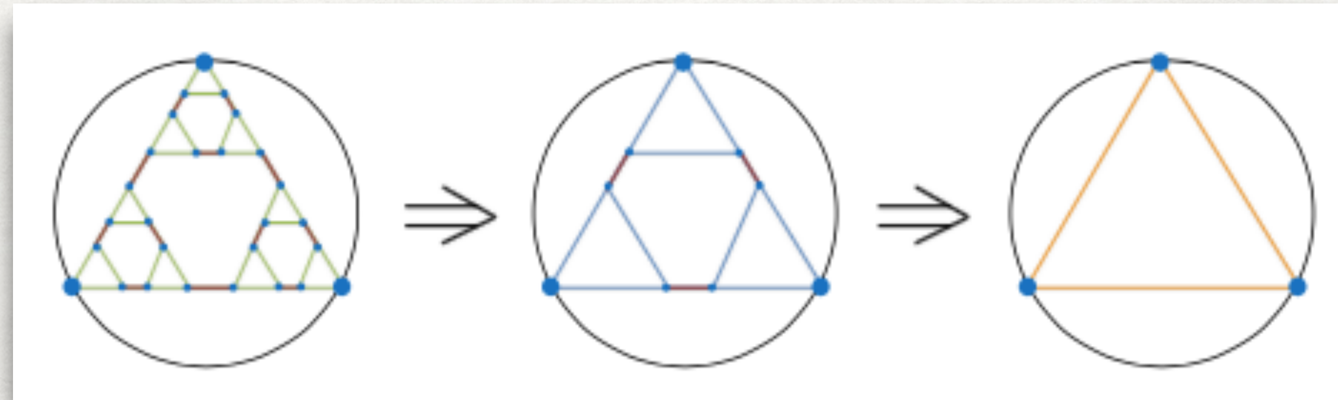
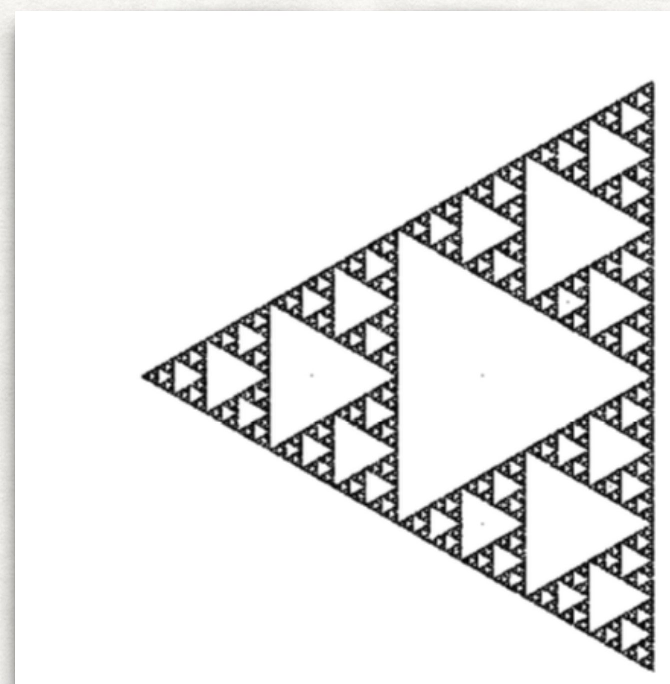


$$c(J_2') = \frac{c(J_1)\,c(J_2)^2}{c(J_2)^2 + 2(1 - c(J_2))(1 + s(J_1) - \frac{1}{2}c(J_1)\,c(J_2))}$$

- Repeating the duality transformation leads to the Sierpinski triangle



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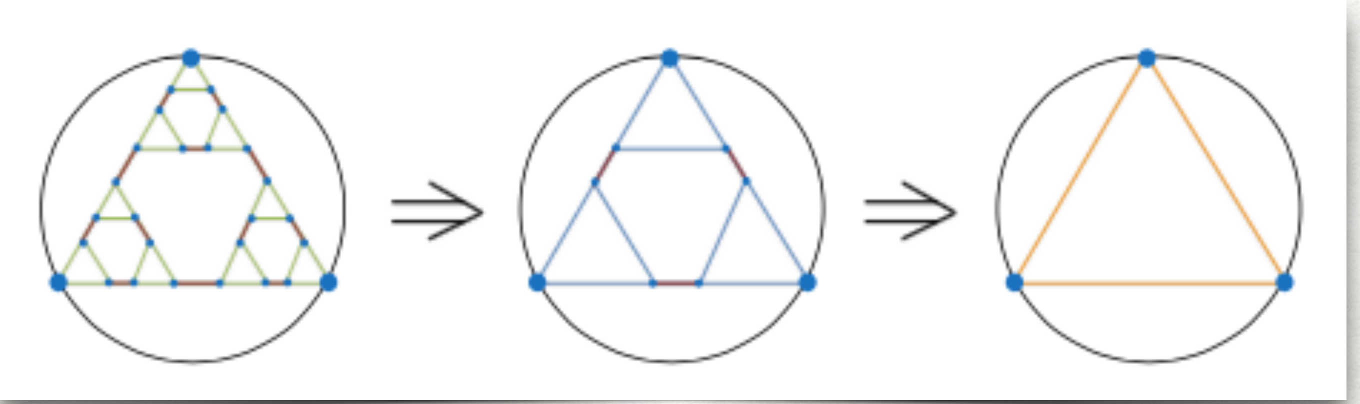

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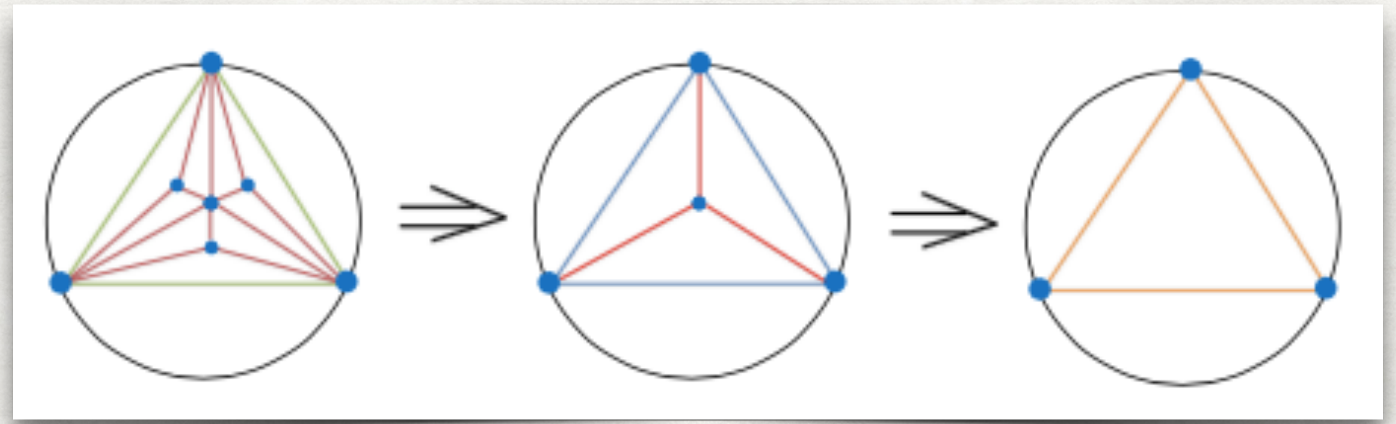
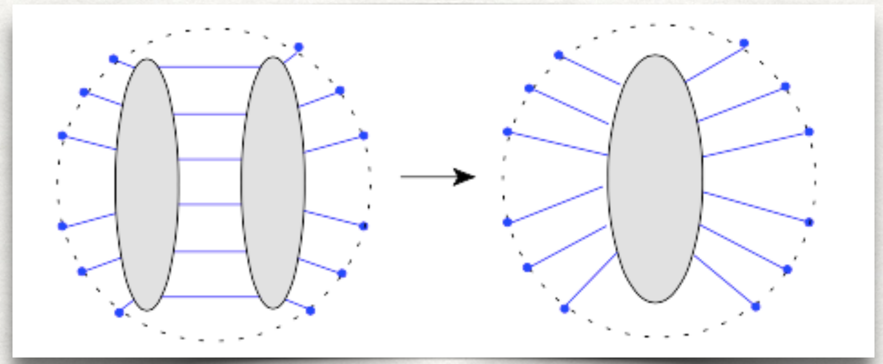


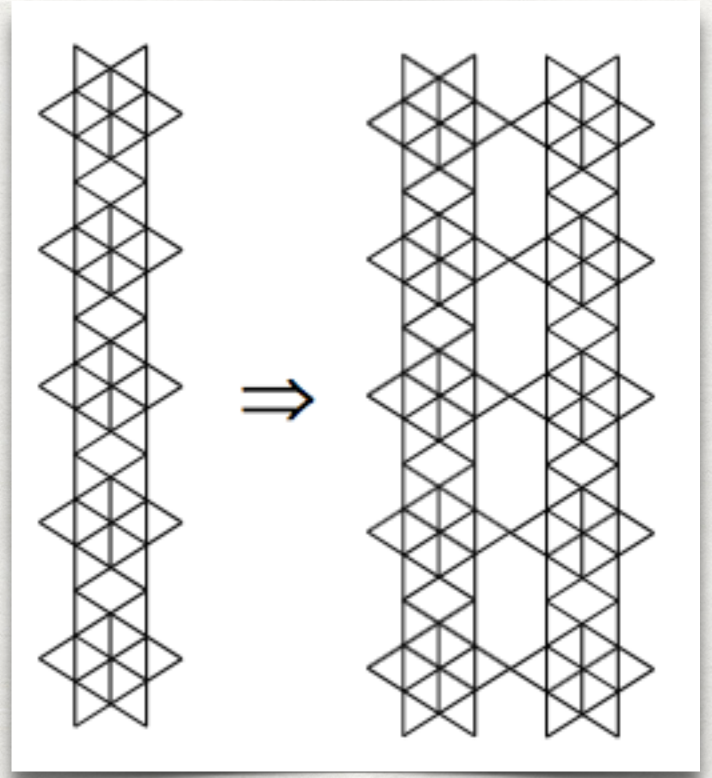
$$c(J') = \frac{1 - s(J_I) + c(J)}{1 + c(J)(1 - s(J_I))}$$

TOTAL POSITIVITY IN ISING

We can also compute the correlator by directly amalgamation in the nx2n matrix!



Start with some nx 2n matrix we simply get another nx2n matrix after amalgamation
we don't get any new complications



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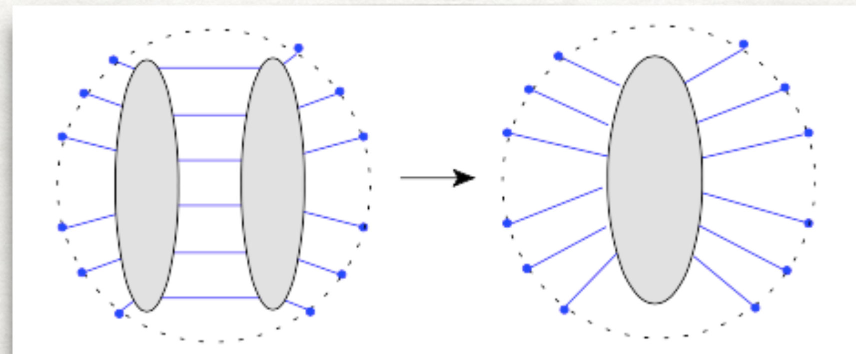
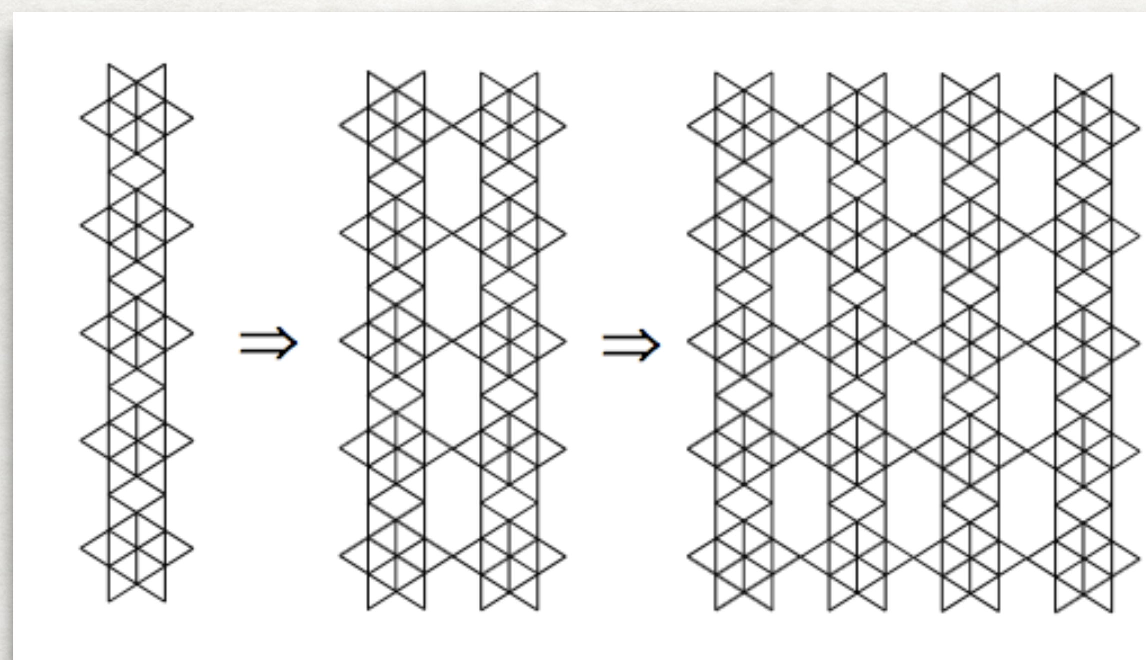
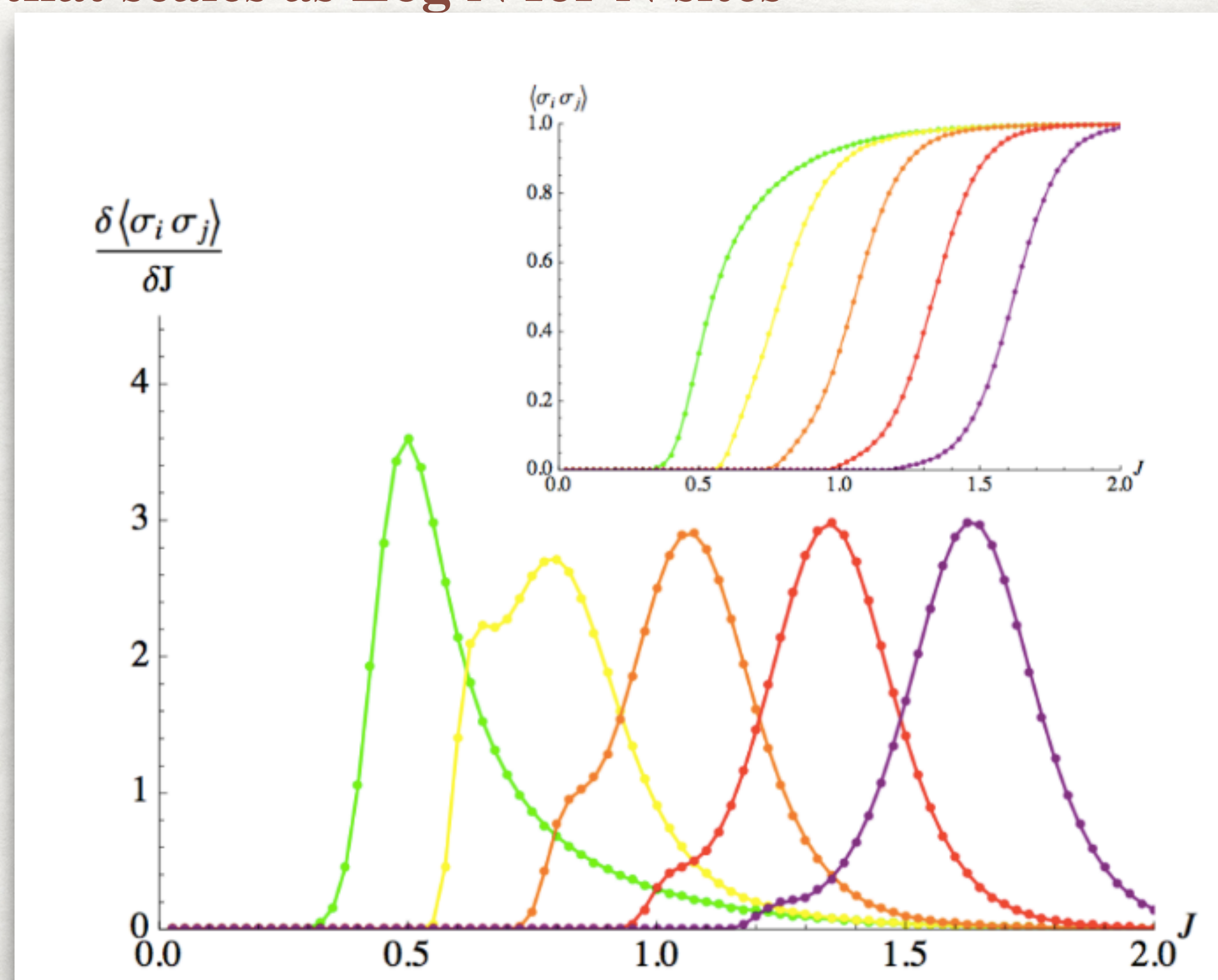


This leads to a computational complexity that scales as Log N for N sites



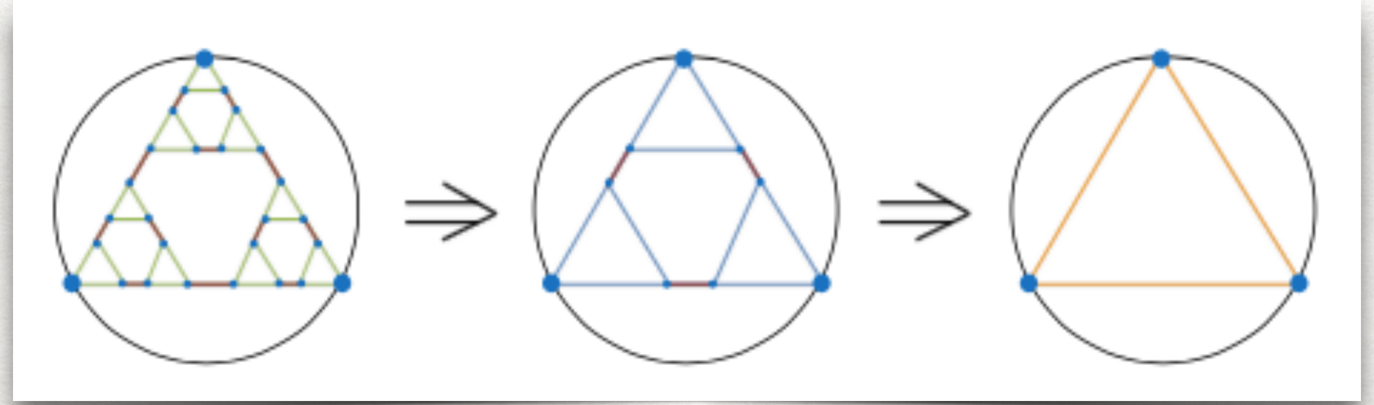


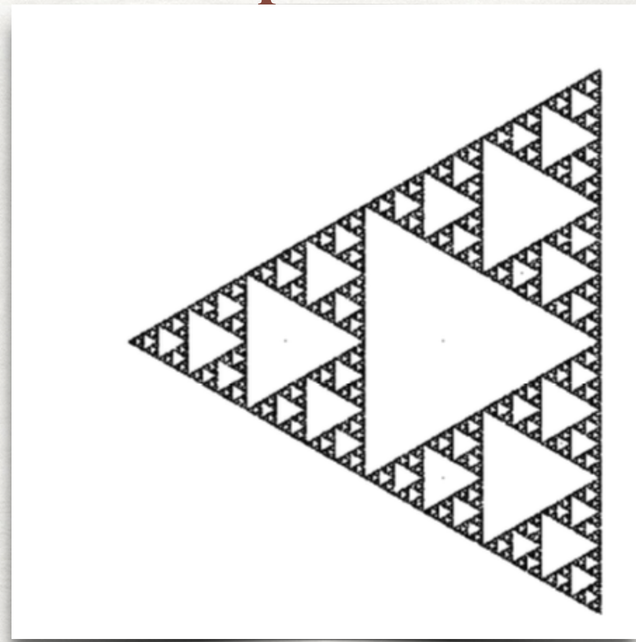
25 iterations results in over

1.7×10^9 spin sites

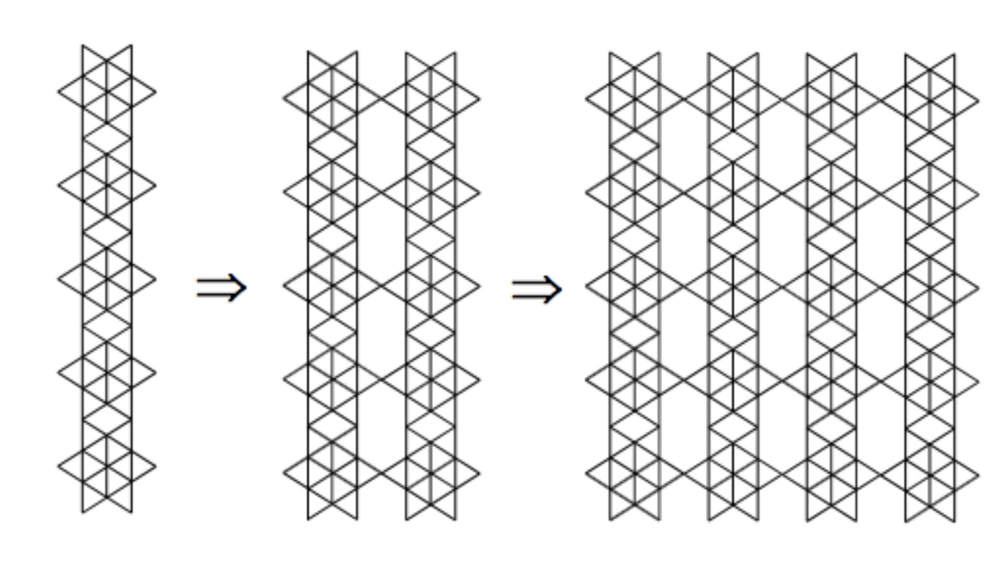
PHASE TRANSITIONS

It is conjectured that lattices with finite ramification number do not exhibit phase transition. The two approaches lead to finite ramification lattices





For fractals constructed from duality transformations, they are dual to 1-d lattice



Amalgamations are simply sums, and the iterated amalgamation simply leads to iterated sum of finite lattices.

SUMMARY

- **The notion of positivity is ubiquitous in physical observables, reflecting the union of physical principles (unitarity, locality, symmetries)**
- **Such positivity is also found for discrete systems such as planar Ising networks. This positivity characterizes topological inequivalent Ising networks. They are organized into equivalent classes under duality moves.**
- **Practical consequences: novel computation methods, and understanding for phase transitions.**
- **Is there hidden positivity behind non-planar, and with external magnetic fields ?**